OPTIMAL BUILDING SHAPE: A GENERAL DESIGN TOOL

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ABSTRACT

Knowing that a building should be "long in the east-west direction" is of limited usefulness unless the designer can answer the question, "How long?" For a rectangular building, the "Building Shape Aspect Ratio" (R) is defined as the building length in the east-west direction divided by the building length in the north-south direction.

Using the Lagrange multiplier optimization technique, an analytic expression is developed to relate the R for energy optimum conditions to various building parameters, such as amount of window glazing, single- or double- pane glazing, building exterior and interior color, degree of shading on a given wall, amount of insulation within frame walls, berming on some sides, etc. Inputs include weather and solar data and proposed architectural features. The technique is reduced to an easily applied design tool, and its usage is generally described as an algorithm that can accommodate well-understood ASHRAE methods. The algorithm is exercised to evaluate R values in different climates for different building parameters, and the effect of different building features on optimum R and overall thermal energy consumption is analyzed.

INTRODUCTION

The ancient Greeks were aware that buildings should be long in the east-west direction to assist in winter heating and also summer cooling. With the modern usage of glass, this wisdom is even more pertinent. For a rectangular building with walls facing the compass points, the building shape aspect ratio (R) is defined as the building length in the east-west direction, x, divided by the building length in the north-south direction, y. Normally, the building area, A = x*y, will be specified, so lengthening x decreases y, and vice versa. The optimum aspect ratio, R, for a considered building in a given winter environment is that R where the minimum thermal energy must be purchased throughout the winter to maintain the building in the thermal comfort zone. Heat losses are augmented by useful solar gain (most of which is incident from the south). In hot climates where summer cooling predominates building thermal design, the optimal R minimizes summer solar gain. The designer makes the judgment on whether the greater environmental stress driving energy consumption occurs during winter or summer. However, it is shown below that building optimal shape tends to be similar in both winter and summer design conditions.

A literature search reveals that little work has been done to optimize R as a function of building wall construction, fenestration quantity and type, interior absorptivity in the solar spectrum, degree of shading on a considered wall, and other factors.

This analysis develops criteria to minimize annual energy expenditure required to maintain the building in the thermal comfort zone. The building type can be rather general. Although the analysis here is outlined in terms of either summer or winter, the algorithm developed can be applied on a basis over a year to obtain an annual optimum.

MATHEMATICAL DEVELOPMENT

For a rectangular building oriented with each of four walls facing a compass point, the floor

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area (and, thus, ceiling and roof area) will be fixed: Area = A = xy. The building length in the east-west direction is x, and y is the building length in the north-south direction. Since the heat loss through the ceiling and floor should be nearly proportional to the floor area, the shape aspect ratio, R = x/y, does not impact on these losses. Also, assume that the infiltration losses or gains are independent of the shape factor. Thus, the only impact we have to consider is the variation of x with y on the environmental losses or gains. Heat losses or gains can occur through walls and through windows, and additional solar gains can occur through windows. For the considered problem, assume that all windows allow a constant percentage of sunlight to pass through them on a daily basis, regardless of angle of incidence. Also, assume that insulating shades or shutters are not applied at night (in winter) or during the day (in summer), although a simple extension of the following analysis would allow such a modification to be made.

Define for each wall (the i-th wall) the Environmental Heating Function, Q_i:

$$\frac{Q_i}{A_i} = \underbrace{f_i \tau\alpha I_i}_{\text{Window solar gain}} - \underbrace{f_i\left(\frac{T_{in}-T_{out}}{R_{win}}\right)}_{\text{Window thermal gain/loss}} - \underbrace{\left(\frac{T_{in}-T_{ext}}{R_{wall}}\right)(1-f_i)}_{\text{Wall thermal gain/loss}} \qquad (1)$$

where

Q_i = Environmental heating function for the i-th wall, the amount of heat per unit area per day (or month) that enters the building through the i-th side; positive signs refer to net gain, negative signs refer to net loss.
$Q = \text{Btu/day}\cdot\text{ft}^2$ or $\text{Btu/month}\cdot\text{ft}^2$ $(\text{kW}\cdot\text{h/day}\cdot\text{m}^2)$

A_i = Area of ith wall, x or y (times wall height)$(\text{ft}^2)(\text{m}^2)$

$\tau\alpha I_i$ = Transmitted and absorbed solar radiation through all windows on the i-th side $(\text{Btu/ft}^2\cdot\text{day})(\text{kW}\cdot\text{h/m}^2\cdot\text{day})$

T_{in} = Building interior temperature, assumed constant throughout a considered period (F)(°C)

T_{out} = Ambient air temperature averaged over a 24 hour period, used for calculating heat gains (other than insolation) or losses through windows (F)(°C)

T_{ext} = Average 24-hour temperature of exterior wall. For a wall shaded from the sun, this can be taken as T_{out}, but for a wall receiving sunlight, this is a calculated quantity and is in excess of T_{out} (F)(°C)

R_{window} = Effective resistance to heat transfer between interior air temperatures and T_{out}, considered over a 24-hour period. If shades or shutters are used to increase resistance during portions of the day, then this must be taken into account (not particularly difficult). $(R = \text{day}\cdot\text{ft}^2\cdot\text{F/Btu})$ $(\text{h}\cdot\text{m}^2\cdot{}^\circ\text{C/ W}\cdot\text{h})$.

R_{wall} = Given (or determined) resistance through an opaque wall. $(R = \text{day}\cdot\text{ft}^2\cdot\text{F/Btu})$ $(\text{h}\cdot\text{m}^2\cdot{}^\circ\text{C/W}\cdot\text{h})$

f_i = Percentage of window area to total window and wall area on the ith side (-)

In the limiting condition where no sun falls on a given wall, then $T_{ext} = T_{out}$ for a good approximation. When the sun srikes the wall, then T_{ext} is a calculated quantity and is greater than T_{out}. The R_{window} parameter is determined from the ASHRAE handbook for a window of appropriate characteristics, as in R_{wall}. During the winter, T_{in} is greater than T_{out}, but generally T_{ext} can be greater OR less than T_{in}, depending on T_{out}, available insolation, and external convective and radiative conductances. Assume no time lag and no heat storage in the walls. In a more detailed analysis, the day can be simulated on an hourly basis for 24 hours. The intent of this paper is to develop and illustrate the technique. A 24 hour heat balance on the outer surface of an opaque wall yields (see Figure 1) T_{ext}.

Energy balance on external wall surface, based on 24 average temperature and conditions, gives:

Heat In = Heat Out

$$\alpha I_i = h_c(T_{ext}-T_{out}) + h_R(T_{ext}-T_{out}) + \frac{T_{ext}-T_{in}}{R_{wall}}$$

Solar gain in wall	Connective loss	Radiative loss	Loss (or gain) through wall to room

Solving for T_{ext} yields

$$T_{ext,i} = \frac{\alpha I_i+(h_c+h_R)T_{out}+T_{in}/R_{wall}}{h_c+h_R + 1/R_{wall}} \tag{2}$$

Wall exterior temperature, 24 average (F)

$h_R = 4\sigma\varepsilon T^3$, linearized radiation heat transfer coefficient (Btu/day$\cdot$ft$^2\cdot$F)

h_c = Conductive heat transfer coefficient (Btu/day$\cdot$ft$^2\cdot$F)

Considering a building side as a composite of windows and opaque walls with fraction, f, of total area in windows, then the expression for Q_i in Equation 1, for a considered building height, becomes

$$Q_i = \left\{\begin{matrix} x \\ \text{or} \\ y \end{matrix}\right\} \left\{ f_i\left[\tau\alpha I_i - \left(\frac{T_{in}-T_{out}}{R_{window}} \right)\right] - (1-f_i)\frac{T_{in}-T_{ext}}{R_{wall}} \right\} \tag{3}$$

Define the following nomenclature conventions (see also Figure 2):

x = length of north and south wall (feet)
y = length of east and west wall (feet)
Positive quantities refer to heat into house

Subscript	Wall	Length
1 = 1	south	x
1 = 2	east	y
1 = 3	west	y
1 = 4	north	x

Now the total heat into or out from a house through the four vertical walls during a considered 24-hour period, assuming all temperatures are averaged constant over the time, is

$$Q = \sum_{i=1}^{4} Q_i \tag{4}$$

During winter and summer, it is assumed that the building is always maintained at a constant temperature through a combination of internal mass storage and HVAC system. Further analysis could relax this assumption. Thus, if during a 24-hour period there is a net heating requirement and sufficient heat enters through windows during the day, overheating is prevented through heat storage, and the excess heat is released as appropriate to put the minimum auxilliary energy requirement on the backup HVAC unit. The same is true in summer; the capability of cooling a storage mass at night for building cooling the following day (night scavenging) is not considered here, although it is a good idea. The only way heat can enter or leave the building is through the exterior skin; however, during winter, active solar collectors could augment the HVAC backup unit. The goal here is to provide the appropriate building aspect ratio, R = x/y, so the Q is maximized in winter, OR Q is minimized in summer, depending on whether summer heat stresses or winter cold stresses should dominate the design. For example, in Phoenix, AZ, summer stresses are more severe than winter stresses; whereas in Minnesota, winter comes with authority. During winter, maximizing Q will minimize auxilliary heat input requirements, providing the optimum condition.

The foregoing equations and this analysis generally consider averaged quantities over a 24-hour period and averaged for a summer or winter season. The more rigorous way to conduct the calculation would be to perform hourly calculations over a 24-hour period for different conditions throughout the year. This could be done using ASHRAE bin weather data or daily meterological data over a considered year. The intent of this paper is to develop the technique and to illustrate the method for representative seasonally averaged winter and summer data.

In Equation 3, all quantities are specified by the designer (or calculated in the case of T_{ext} from Equation 2), except for the wall lengths x or y, which are to be optimized. But the optimization of Equation 4 is subject to the area constraint on x and y, namely

$$A = x * y, \text{ specified and constant CONSTRAINT} \tag{5}$$

$$\text{or constant } \emptyset = xy - A = 0$$

So, increasing x decreases y, and vice versa; their product is constant and specified by the designer, who wants a building of a given floor area. Since we are optimizing the objective function, Equation 4, with a single constraint, Equation 5, the technique of Lagrange multipliers is appropriate. Generally, each constraint equation gives rise to a Lagrange multiplier (Stoecker 1980). Here, there is a single constraint equation, so there will be only one Lagrange multiplier, λ. The well known formulation is, for a single constraint

$$\nabla (Q - \lambda \emptyset) = 0$$

where ∇ is the "del operator." In two dimensions

$$\nabla z = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Then, since Q = Q(x,y), and $\emptyset = \emptyset(x,y)$, we get

$$\frac{\partial Q}{\partial x} - \frac{\partial \emptyset}{\partial x} = 0 \tag{6}$$

and

$$\frac{\partial Q}{\partial y} - \lambda \frac{\partial \emptyset}{\partial y} = 0 \tag{7}$$

Equations 5, 6, and 7 provide three equations for the three unknowns, which are x, y, and . The solution gives x* and y*, where the asterisk means optimal dimension.

Let us now apply the above equations to develop a somewhat general expression for x* and y*. We shall also see that λ is a useful and interesting parameter, being a sensitivity coefficient describing additional heat requirement for a change in area (A = xy) near the solution. In Equation 2, we see that T_{ext} is a calculated and knowable value, independent of the unknowns x and y, and is, thus, predetermined from given design data. Therefore, in Equations 3 and 4, the only unknowns are x and y. We then get from Equations 6 and 7

$$\frac{\partial Q}{\partial x} - \lambda \frac{\partial \emptyset}{\partial x} = 0$$

or

$$f_1 \left[\zeta \alpha_1 I_1 - \left(\frac{T_{in} - T_{ext}}{R_{window}} \right) \right] - (1 - f_1) \left[\frac{T_{in} - T_{ext,1}}{R_{wall}} \right]$$

$$+ f_4 \left[\zeta \alpha_4 I_4 - \left(\frac{T_{in} - T_{out}}{R_{window}} \right) \right] - (1 - f_4) \left[\frac{T_{in} - T_{ext,4}}{R_{wall,4}} \right] - \lambda y = 0 \tag{8}$$

Thus, for given values of f_1, f_4, etc., $\partial Q / \partial x$ can be evaluated. Similarily, Equation 7 becomes

$$\frac{\partial Q}{\partial y} - \lambda x = 0 \tag{9}$$

where

$$\frac{\partial Q}{\partial y} = f_2\left[\zeta\alpha I_2 - \left(\frac{T_{in}-T_{out}}{R_{window}}\right)\right] - (1-f_2)\frac{T_{in}-T_{ext,2}}{R_{wall,2}}$$

$$+ f_3\left[\zeta\alpha I_3 - \left(\frac{T_{in}-T_{ext}}{R_{window}}\right)\right] - (1-f_3)\frac{T_{in}-T_{ext,3}}{R_{wall,3}}$$

Eliminating λ from Equations 8 and 9 gives the sought after Building Optimal Shape Aspect Ratio, R = x/y, or

$$r = \frac{x}{y} = \frac{\frac{\partial Q}{\partial y}}{\frac{\partial Q}{\partial x}} \tag{10}$$

The total amount of heat gain or loss from the four building walls is also of interest in making design comparisons. Comparing Equations 3 and 8, we see that

$$Q = \sum_{i=1}^{4} Q_i$$

and

$$Q = 2x\frac{\partial Q}{\partial x} + 2y\frac{\partial Q}{\partial y} \tag{11}$$

and 10 is R = x/y. Plug 5 into 11 to eliminate y, then insert 10 into 11 to get

$$Q = 2\sqrt{AR}\left[\frac{\partial Q}{\partial x} + \frac{1}{R}\,\frac{\partial Q}{\partial y}\right] \tag{12}$$

But from Equation 10, the second term in the bracket of Equation 12 is equal to $Q/\ x$, and so we get

$$Q = 4\sqrt{AR}\,\frac{\partial Q}{\partial x} \tag{13}$$

Note the equality of the two terms in the bracket in Equation 12. This condition of heat loss equality from the two walls in the x-direction with the two y-direction walls appears to be consistent with size restriction in the direction of greatest heat loss. That is, if a side or sides have relatively high heat loss, then the linear distance allocated to that dimension will be restricted in a proportional manner, resulting in equal heat loss from the x and y axis walls. The square root relationship between Q and floor area A is not surprising, because the heat losses considered here are proportional to wall area, which is proportional to perimenter, which, in turn, would be proportional to square root of the floor area A.

ILLUSTRATIONS OF APPLICATION

Winter Condition

For illustration we will use January data for Reno, NV. Reno is at 40°N latitude. January averages about 1000 degree-days, typical of many locations at 40°N. A typical January day averages 32 F over a 24-hour period. From Berdahl et al. (1978) we get the following insolation schedule, reported here as Table 1.

Consider cases where walls have a thermal resistance value of R-20 and windows are thermopane with double glazing. The ASHRAE Handbook (1985) gives, for thermopane windows at winter conditions, an R value of $R_{window} = 1.6\ h\cdot ft^2\cdot F/Btu$. For a south-facing wall, we can directly calculate T_{ext} from Equation 2. The ASHRAE Handbook recommends for winter conditions with winds of 15 mph (6.7 m/s) combined radiation and convection coefficient $(h_c + h_R) = 6\ Btu/h\cdot ft^2\cdot F$, and for summer with a wind velocity of only half, or 7.5 mph, a combined coefficient of $4\ Btu/h\cdot ft^2\cdot F$. But the ASHRAE data are designed to help size HVAC equipment to provide heating for the most severe condition likely to be encountered, whereas we are concerned with a daily or even monthly average condition because we are considering environmental gain or loss with full backup HVAC capability. Since 15 mph sounds like an excessive sustained 24-hour-per-day wind velocity and 7.5 mph sounds more reasonable, let us use a combined heat transfer coefficient of $(h_c + h_R) = 4.0\ \text{Btu/h·ft}^2\text{·F}$. In some cases, it may be necessary to calculate h_R separately from h_c (h_R has a value of approximately $0.68\ \text{Btu/h·ft}^2\text{·F}$), but that is not necessary here. For a wall solar absorptivity of 0.8 and interior air temperature $T_{in} = 70$ F, Equation 2 gives for the south vertical wall (wall 1)

$$T_{ext,1} = \frac{\left(\frac{1\ \text{day}}{24\ \text{h}}\right)(0.8)\left(1489\ \frac{\text{Btu}}{\text{ft}^2\text{day}}\right)+\left(4\ \frac{\text{Btu}}{\text{h·ft}^2\text{·F}}\right)(32\ \text{F}) + \frac{70\ \text{F}}{20\ \text{h·ft}^2\text{·F/Btu}}}{(4\ \text{Btu/h·ft}^2\text{·F}) + \frac{1\ \text{Btu}}{20\ \text{h·ft}^2\text{·f}}} = \frac{49.6+128+3.5}{4.05} = 44.7\text{F}$$

$$T_{1,ext} = 44.7\ \text{F}$$

A similar calculation for the east and west walls, which have symmetrical incident insolation equal to $517\ \text{Btu/ft}^2\text{·day}$, gives

$$T_{ext,2} = T_{ext,3} = 36.7\ \text{F}$$

and the north wall (wall 4), which receives diffuse solar radiation in the quantity $I_4 = 119\ \text{Btu/day·ft}^2$, has an average exterior wall temperature of

$$T_{ext,4} = 33.4\ \text{F}$$

Assume all double-pane windows have a total solar transmittance of $(0.9)^2 = 0.81$ and the effective interior solar wavelength absorptivity is 0.86, then the fraction of sunlight incident on a given window that is usefully absorbed inside the building is

$$\mathcal{T}^2\alpha = (0.9)^2(0.86) = 0.70$$

Case 1: All glass wall building. Consider an office building where all walls are glass (double-pane thermopane). Then all the $f_i = 1.0$. From Equation 8

$$\frac{\partial Q}{\partial x} = (1.0)\ \frac{0.7(1489)}{24\ \text{h/day}} - \frac{(70-32\ \text{F})}{1.6} + (1.0)\ \frac{(0.7)(119)}{24} - 23.75$$

$$= 43.43-23.75 + 3.47-23.75 = 19.68-20.28 = -0.60$$

$$\frac{\partial Q}{\partial x} = -0.6\ \text{Btu/h·ft}^2$$

Similarly, using Equation 9 with symmetrical solar inputs

$$\frac{\partial Q}{\partial x} = (1.0)\ \frac{0.7(517)}{24} - 23.75 \quad x\ 2 = 2 \quad 15.08 - 23.75$$

$$\frac{\partial Q}{\partial y} = -17.34\ \text{Btu/h·ft}^2$$

Then from Equation 10 we get

$$R = \frac{x}{y} = \frac{\frac{\partial Q}{\partial y}}{\frac{\partial Q}{\partial x}} = \frac{-17.34}{-0.6} = 28.9$$

$$Q = 4\sqrt{AR}\quad \frac{\partial Q}{\partial x} = 4\ (28.9)(-0.6)\ \sqrt{A}$$

$$Q = -12.9\sqrt{A}\ \text{Btu/h·ft}^2\text{·floor·area}$$

This indicates for the assumptions and conditions considered that such a glass-walled building should be 28.9 times as long in the east-west axis as it is wide. This is a consequence that the combined north and south walls are only marginally an energy loser (-0.6 Btu/h·ft^2); whereas, the east and west walls lose much more energy to the cold environment than they gain by solar heating. Also note that with some insulation on the north side, the combined north and south walls could have been a net energy gainer, in which case R above would have been negative! The interpretation of a zero or negative R is that the optimal mathematical aspect ratio is infinite because the long direction is a net energy gainer and the short side is a net energy loser, indicating that the short side would be best at zero dimension. In practice, practical considerations would limit the sizes. For example, the cost of a wall rises as the wall area increases, even though the floor area remains constant. Still, for a glass office building in a cold climate, it is clear that it should be as long in the east-west axis as practical from the energy-conservation standpoint. All results are assembled in Table 2 for easy comparison.

Case 2. No windows in the building and all walls R-20. For Case 2, all $f_i = 0$. Then Equations 8 and 9 give

$$\frac{\partial Q}{\partial x} = -\overbrace{(1.0)}^{1-f_1}\left[\frac{70\text{ F}-44.7\text{ F}}{20}\right] - \overbrace{(1.0)}^{1-f_4}\left[\frac{70-33.4}{20}\right] = -1.265 - 1.83$$

$$\frac{\partial Q}{\partial x} = 3.095\ \text{Btu/h·ft}^2$$

$$\frac{\partial Q}{\partial y} = -2\,\frac{70-36.7}{20} = -3.33\ \text{Btu/h·ft}^2$$

so

$$R = \frac{x^*}{y^*} = \frac{\frac{\partial Q}{\partial y}}{\frac{\partial Q}{\partial x}} = \frac{-3.33}{-3.095} = 1.07$$

$$Q = 4\sqrt{AR}\,\frac{\partial Q}{\partial x} = -12.8\sqrt{A}\ \text{Btu/h·ft}^2\text{·floor area}$$

The no-window case produces an almost square structure, only 7% longer in the east-west direction than wide, in marked contrast to the very long aspect ratio associated in Case 1 with the all-glass building. This is a consequence that all walls perform nearly equally, giving little directional preference to orientation.

Case 3. 40% windows on south side, 20% on other three sides.

$$\frac{\partial Q}{\partial x} = \overbrace{(0.4)}^{f_1}(19.68) + \overbrace{(0.6)}^{1-f_1}(-1.265) + \overbrace{(0.2)}^{f_4}(-20.28) + \overbrace{(0.8)}^{1-f_4}(-1.83)$$

$$\frac{\partial Q}{\partial x} = +\,1.593\ \text{Btu/h·ft}^2$$

The positive value means for the considered conditions that the north-south wall combination is a net energy gainer, a direct consequence of the relatively large south-facing window, which allows direct solar gain.

$$\frac{\partial Q}{\partial y} = 2\left[\underbrace{(0.27)(-8.67)}_{\text{windows}} + \underbrace{(0.8)(-3.33)}_{\text{walls}}\right] = -8.8$$

$$R = \frac{\frac{\partial Q}{\partial y}}{\frac{\partial Q}{\partial x}} = \frac{-8.8}{+1.593} = -5.52$$

$Q = \sqrt{4\ \ AR}$ $\frac{\partial Q}{\partial x}$ becomes sparious in view of negative R

The negative value for R means that the building should be infinitely long in the east-west axis, since the south side and north side combination provide a net heat gain, whereas the east-west sides give a net loss.

Case 3a. 20% windows on all sides. Now assess the impact of a south-facing window area. Reduce the 40% south window area in Case 3 to 20%, which will be Case 3a.

$$\frac{\partial Q}{\partial x} = (0.2)[19.68] + (0.8)[-1.265] - 5.52 = -3.6 - 1.012 - 5.52 = 3.6$$

$$\frac{\partial Q}{\partial y} = -8.8 \text{ from Case 3}$$

$$R = \frac{-8.8}{-3.6} = 2.44$$

Comparison with Case 3 shows the advantage of maximizing south side windows as much as possible.

Case 3b. No windows on the south side, 20% windows on all other sides. To further illustrate the importance of south-facing windows, remove all windows from the south side and replace them with R-20 walls. Leave 20% windows on the other three sides; this will be Case 3b. The above technique yields

$$R = \frac{-8.8}{-6.785} = 1.3$$

Previously, in Cases 3 and 3a, windows on the south side had made the south side the only energy-gaining side due to available insolation. Shutting up south side windows and replacing them with a wall makes the south side a net energy loser like all the other sides, and the impact on building shape is to scrunch the building more nearly into a square (R = 1.3). Thus, we clearly see that south-facing windows in this climate open up the design long in the east-west direction, and removal of windows condenses the floor plan to a more nearly square shape.

Case 3c. 20% windows on the north side, no windows on the other side. Suppose an owner has a particularily fine view to the north and wishes to open his building window-wise to the north, but to close it up on the other sides to reduce heat losses. (Recall east and west windows are net heat losers compared to R-20 east and west walls.) In this scernario, there might be a junk view to the south (which still recives full sunlight) to motivate replacement of south windows by R-20 wall. Now we get

$$\frac{\partial Q}{\partial x} = -6.785, \text{ as in Case 3b}$$

$$\frac{\partial Q}{\partial y} = -6.66$$

$$R = \frac{\frac{\partial Q}{\partial y}}{\frac{\partial Q}{\partial x}} = \frac{-6.66}{-6.785} = 0.98$$

Here we have a case where the building is slightly longer in the north-south axis than the east-west direction in an attempt to reduce the north side window area, a major source of loss. Note that the south wall still performs better than the east or north walls, since the effective 24-hour external temperature of the sunlit south wall is 44.7 F, compared to 36.7 F for the east and west walls, and 33.4 F for the north wall.

Case 4. Single-pane windows. Instead of the double-glazed thermopane window previously considered, consider now single-pane windows and assess the difference such windows make on optimal building shape. The single-pane window allows more sunlight to be transmitted but also allows more heat to be lost. ASHRAE (1985) recommends a single-pane window for winter condition $R = 0.88\ h\cdot ft^2\cdot F/Btu$. Then $\tau\alpha = (0.9)(0.86) = 0.774$. For the all-glass building (f = 1.0)(Case 4a)

$$\frac{\partial Q}{\partial x} = \underbrace{\left[\overset{48.0}{\frac{0.774(1489)}{24}} - \overset{-43.2}{\frac{-70-32}{0.88}}\right]}_{\substack{\text{south side} \\ +4.8}} + \underbrace{\left[\overset{3.83}{\frac{0.774(119)}{24}} - 43.2\right]}_{\substack{\text{north side} \\ -39.34}}$$

$$\frac{\partial Q}{\partial x} = -34.5 \text{ Btu/h·ft}^2$$

and

$$\frac{\partial Q}{y} = -53$$

$$R = \frac{-53}{-34.5} = 1.54$$

$$Q = 4\sqrt{1.54}\,(-34.5)\sqrt{A} = -171.3\sqrt{A} \text{ Btu/h·ft}^2$$

For the thermopane all-glass building, the R-ratio was 28.9, which for practical purposes could have been infinite. The increased losses arising from the single pane (Case 4a) means all sides lose much more heat, and the advantages of south side windows are reduced, compressing the ideal shape into a more nearly square shape (R = 1.54).

Consider now a Case 4b, which is the same as Case 3 with 40% windows on the south side and 20% windows on the other three sides. All walls are R-20; however, only single-pane windows are used. Now going through the numbers, as above, gives

$$R = \frac{-13.26}{-8.171} = 1.62$$

This is to be compared with the R of Case 3, which had a negative value indicating that an infinitely long structure would be appropriate from the standpoint of minimum building energy consumption. The conclusion is that storm windows, thermopane glass, and insulating shutters deployed at night all significantly reduce heat loss when the sun is not shining and elongates the optimum building shape in the east-west direction.

Case 4c is Case 3a, which has 20% windows on all sides; however, only single-pane glass is specified. The numbers give

$$R = \frac{-13.26}{-9.384} = 1.41$$

$$Q = -44.6\sqrt{A} \text{ Btu/h·ft}^2 \text{ floor area}$$

Thus, we conclude that insulated windows allow the building shape to expand in the east-west direction.

Case 5. Light colored interior. Previous cases have assumed that sunlight entering through the windows is mostly absorbed inside the room and converted to heat. The absorption factor was assumed to be 0.86. But suppose the interior paint is very light, so less solar heat is absorbed by the walls and, consequently, reflected back out the window. Suppose the effective interior absorption coefficient is 0.5, instead of the previously assumed 0.86, then

$$\tau^2\alpha = (0.9)^2(0.5) = 0.49$$

instead of $\tau^2\alpha = 0.70$. Then the all-double-pane-glass building (Case 1) reduces the R value to

$$R = \frac{\frac{\partial Q}{\partial y}}{\frac{\partial Q}{\partial x}} = \frac{-26.4}{-14.67} = 1.8$$

which is way down from the dark interior wall value of R = 28.9 in Case 1. The obvious conclusion is that the color of interior finish has a definite impact on the optimum building shape. Since dark interior finishes absorb more incoming sunlight, they will be warmer, effectively producing more heat rather than reflecting it back out the window, and less backup energy will be required to maintain the structure within the thermal comfort zone. Although dark interior finish shows a benefit during winter and stretches out the value of R, it may not be best for summer; this remains to be determined.

Case 6. Shaded south wall, 20% windows on all walls. What happens to the building shape if certain walls are shaded, especially the south wall? Actually, shading would likely imply some protection from the wind, so there can be a slight benefit from shading that would not affect the large disadvantage arising from decreased solar gain. Wind breaks on the north side would be a positive help (and possibly on the east and west sides) to reduce convective and radiative heat losses. Naturally, shading of south-facing windows will reduce or eliminate possibility of solar gain, a disadvantage. Consider Case 3a, which had 20% windows on all sides, but assume the south side is shaded so it thermally behaves like the north side, receiving only small amounts of diffuse sunlight. East and west walls and windows are not shaded and receive full possible sunlight. Then

$$\frac{\partial Q}{\partial y} = -8.8, \text{ as in Case 3a}$$

$$\frac{\partial Q}{\partial x} = 2[-5.52] = -11.4$$

so

$$R = \frac{x}{y} = \frac{\frac{\partial Q}{\partial y}}{\frac{\partial Q}{\partial x}} = \frac{-8.8}{-11.04} = 0.80$$

Since the building cannot receive the normal benefits of a southern exposure due to shading, it wants to open up its east and west walls by expanding the north-south axis direction.

Case 7. Reduced insulation in walls (R-7). Consider the impact on building shape of less insulation in walls. Specifically, consider Cases 3 and 3a with R-7 walls instead of R-20. With wall thermal resistance changed, the equilibrium daily averaged external wall temperatures will also change and need to be recalculated, using Equation 2. With R-7 and winter conditions

Wall	I_i $\frac{\text{Btu}}{\text{day}\cdot\text{ft}^2}$	$T_{ext,i}$
1 south	1489	45.3 F
2/3 east/west	517	37.5 F
4 north	119	34.3 F

Case 7 resembles Case 3, with 40% thermopane windows on the south side and 20% windows on the other three sides.

$$R = \frac{\frac{\partial Q}{\partial y}}{\frac{\partial Q}{\partial x}} = \frac{-10.9}{-2.39} = 4.56$$

Case 7a. 20% windows on all sides, R-7 in walls.

$$R = \frac{-10.9}{-7.03} = 1.55$$

Summer Condition

For the summer condition, use an average 24-hour air temperature of 76 F. Also, from Table 1 use the given insolation data for the four walls. It is stressed that the illustration here averages temperature and insolation conditions to 24 hours. For more detail, the calculation should likely be taken on an hourly basis, especially in the summer when insolation values and hours are greater than in winter. Also, a second generation calculation might consider variation of temperature within the building throughout the day, as well as night scavenging. The examples here merely illustrated the techniques. The analysis previously developed and illustrated for winter condition also applies for the summer condition. Using $(h_c+h_R) = 4.0$ $\text{Btu/h}\cdot\text{ft}^2\cdot\text{F}$, wall absorptivity = 0.8, R-20 for the wall, $T_{out} = 76$ F, and $T_{in} = 70$ F, we get from Equation 2

$$\text{South } T_{ext} = 82.8 \text{ F}$$

$$\text{East/West } T_{ext} = 86.4 \text{ F}$$

$$\text{North } T_{ext} = 79.5 \text{ F}$$

Then we can plug inputs into the appropriate equations, as was illustrated for the winter condition. The results for some cases are given in Table 2.

Case 1. All windows, summer condition.

$$\frac{\partial Q}{\partial x} = \left[\frac{0.7(839)}{24} - \frac{70-76}{1.6}\right] + \left[\frac{0.7(430)}{24} + 3.75\right]$$

$$= \underset{\text{south}}{28.2} + \underset{\text{north}}{16.3}$$

$$\frac{\partial Q}{\partial x} = +44.5 \text{ Btu/h·ft}^2\text{·floor area}$$

and

$$\frac{\partial Q}{\partial y} = 82 \qquad R = \frac{82}{44.5} = 1.84$$

$$Q = 4\sqrt{AR}\ \frac{\partial Q}{\partial x} = 4\sqrt{1.84}\,[44.5]\sqrt{A}$$

$$Q = 241.6\sqrt{A} \text{ Btu/h·ft}^2\text{·floor area}$$

All those windows give rise to an egregious heat gain, which will require an enormous air-conditioning load. Since there is more sunlight on east and west windows than on the north and south, the building elongates itself (in the optimal summer shape) to escape the morning east sun and afternoon west sun. Again, the elongation is along the east-west axis, just as during the winter.

Case 2. All R-20 walls, summer condition.

$$\frac{\partial Q}{\partial x} = +1.12$$

$$\frac{\partial Q}{\partial y} = +1.64$$

$$R = \frac{1.64}{1.12} = 1.46$$

$$Q = 4\sqrt{1.46}\ [1.12]\sqrt{A} = 5.42\sqrt{A} \text{ Btu/h·ft}^2$$

Some of the cases analyzed for winter in Table 2 are also considered for the summer condition in Table 2; further calculations are not shown, only the results.

CONCLUSION

An algorithm has been developed that allows determination of the optimum wall dimensions (for a given floor area), leading to the minimum energy consumption consistent with the climate and building characteristics. The technique is illustrated here for both a winter and summer condition for various building conditions. However, only monthly averaged environmental data are used here for the examples. An in-depth calculation would probably require hourly calculations over a 24-hour period. Also, the in-depth calculation would require similar calculations for each of 12 months, with the results being subjected to some kind of integration or averaging to obtain annual optimal dimensions.

The summer and winter averaged data used here do yield some interesting tentative quantitative results. These results are summarized in Table 2. The impact on optimal building shape is shown to be dependent upon number of glazings for each window, insulation R rating in walls, solar absorptivity behind a window, south side shading in winter, east and west side shading in summer, percentage of windows along a given wall, and climate.

REFERENCES

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TABLE 1
Incident Total Insolation, Reno (40°N)
$Btu/ft^2 \cdot day$

Wall	Winter	Summer
1 south	1489	839
2 east	517	1277
3 west	517	1277
4 north	119	430
horizontal	787	2619

TABLE 2
Building Shape Aspect Ratio and Heat Requirements for Different Conditions

Winter Condition: T_{out} = 32 F average 24-hour temperature
Sunny day at latitude 40°N

		Winter Condition		Summer Condition	
CASE	DESCRIPTION	SHAPE ASPECT RATIO $R=x^*/y^*$	BUILDING HEAT REQUIREMENT Q ($Btu/h \cdot ft^2 \cdot floor$) HEAT LOSS	R	Q ($Btu/h \cdot ft^2 \cdot floor$) HEAT GAIN
1	All glass building, thermopane 2-glass	28.90	-12.9	1.84	241.0
4a	All glass building, single-pane glass	1.54	-171.0		
2	All R-20 walls, no windows	1.07	-12.8	1.46	5.42
3	40% glass on s-side, 20% glass elsewhere	-5.50		1.16	65.8
3b	No s-side windows, 20% windows elsewhere	1.30	-30.9	4.14	33.7
3c	20% windows on n-side, no windows elsewhere	0.98	-26.9	0.31	9.5
4b	Same as Case 3, but single-pane windows	1.62	-41.6		
4c	Same as Case 3a, but single-pane windows	1.41	-44.6		
5	All glass (Case 1), but light interior	1.80	-78.7		
6	20% windows (Case 3a), but s-side shaded	0.80	-39.5	3.60	37.3
7	Like Case 3, but R-7 walls	4.56	-20.4		
7a	Like Case 3a, but R-7 walls	1.55	-35.0		

NOTE: (1) Assumes the building is unshaded unless otherwise specified. Possible snow relection not considered.
(2) All windows are assumed to be double-glass thermopane (R-1.6) unless otherwise specified. Single pane have R-0.88.
(3) All walls are assumed to have R-20 thermal resistance, except for Case 7.

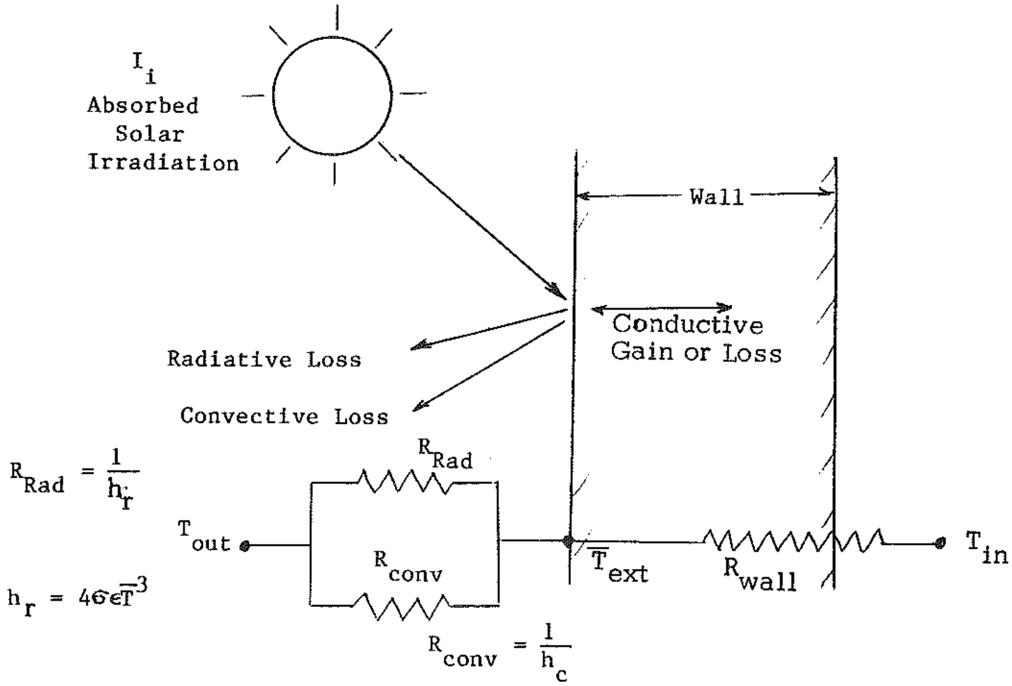

Figure 1. Illustrating heat balance leading to Equation 2

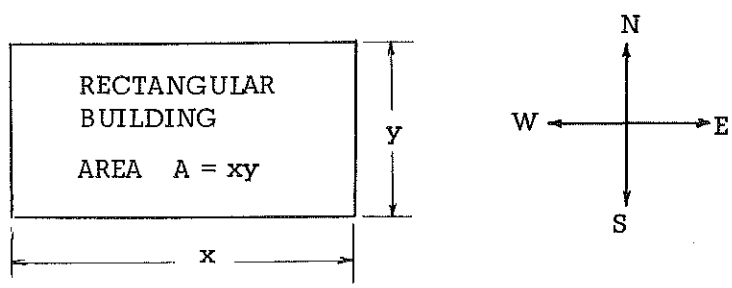

Figure 2. Building shape aspect ration, R: R = x/y; x = length of north and south wall (ft); y = length of east and west wall (ft). Positive heat quantities refer to heat in house

Subscript	*Wall*	*Length*
$i = 1$	*South*	x
$i = 2$	*East*	y
$i = 3$	*West*	y
$i = 4$	*North*	x